

Relativistic Coulomb Sum Rules for (e, e')

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Abstract

A Coulomb sum rule is derived for the response of nuclei to (e, e') scattering with large three-momentum transfers. Unlike the nonrelativistic formulation, the relativistic Coulomb sum is restricted to spacelike four-momenta for the most direct connection with experiments; an immediate consequence is that excitations involving antinucleons, e.g., $N\bar{N}$ pair production, are approximately eliminated from the sum rule. Relativistic recoil and Fermi motion of target nucleons are correctly incorporated. The sum rule decomposes into one- and two-body parts, with correlation information in the second. The one-body part requires information on the nucleon momentum distribution function, which is incorporated by a moment expansion method. The sum rule given through the second moment (RCSR-II) is tested in the Fermi gas model, and is shown to be sufficiently accurate for applications to data.

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1 Introduction

Electrons have proven to be useful probes for the study of nuclear structure, primarily because the eN interaction is reasonably well-known and because the electromagnetic interaction is weak compared to the strong forces which dominate nuclear structure. The longitudinal contribution to the nuclear response is of particular interest because of its relationship to two-body correlation functions through what is commonly known as the Coulomb sum rule (CSR).[1] With the assumption that the nucleus may be treated as a nonrelativistic system of interacting nucleons, the extraction of nucleon-nucleon correlations is formally direct: this is commonly referred to as the nonrelativistic Coulomb sum rule (NRCSR). In practice, however, one finds that to obtain useful information one must extend the experiments and analysis to energies and momentum transfers sufficiently large that the usual nonrelativistic assumptions fail. This will be particularly the case for experiments at the Continuous Electron Beam Accelerator Facility (CEBAF) with $E_{\text{beam}} \simeq 4$ GeV. Therefore, it has become desirable to have a fully-relativistic formalism both for analyzing experimental data and for testing theoretical models of nuclear structure. In this paper, we present an extension of the NRCSR which successfully handles certain problems which arise in electron scattering from nuclei at large three-momentum transfers.

Before discussing further the goals of this paper, it will be useful to review the form of the NRCSR, as well as its derivation. We begin with the first-order Born (or one-photon exchange) approximation for the scattering of ultrarelativistic electrons ($|\mathbf{k}| \gg m_e$) from nuclear targets, in which one can factor the (e, e') differential cross-section into leptonic and nuclear parts. After performing a standard separation of this nuclear response function into longitudinal and transverse (virtual) photon contributions, the differential cross-section in the laboratory frame can be written

$$\frac{d^2\sigma}{d\Omega' dE'} = \frac{d\sigma}{d\Omega'} \Big|_{\text{Mott}} \left[\frac{Q^4}{\mathbf{q}^4} W_C(\omega, \mathbf{q}) + \left(\frac{1}{2} \frac{Q^2}{\mathbf{q}^2} + \tan^2 \frac{\theta}{2} \right) W_T(\omega, \mathbf{q}) \right], \quad (1.1)$$

where the Mott cross-section

$$\frac{d\sigma}{d\Omega'} \Big|_{\text{Mott}} \equiv \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \quad (1.2)$$

describes the elastic scattering of ultrarelativistic electrons from a *fixed* point target with charge e and no spin. Here $q^\mu = (\omega, \mathbf{q})$ is the four-momentum transfer to the target, θ is the electron scattering angle, E is the electron beam energy and $Q^2 \equiv -q^2 = \mathbf{q}^2 - \omega^2$. The first term in the square brackets in (1.1) is the longitudinal contribution, and is usually expressed

in terms of the Coulomb response function:

$$W_C(\omega, \mathbf{q}) \equiv \sum_f |\langle f | \hat{\rho}(q) | i \rangle|^2 \delta(\omega - E_f + E_i), \quad (1.3)$$

where $|i\rangle$ and $|f\rangle$ denote initial¹ and final nuclear states, respectively, and $\hat{\rho}(q)$ is the Fourier transform of the nuclear charge density operator.

In general, the operator $\hat{\rho}(q)$ should include contributions from both the nucleons and (virtual) charged mesons in the target, but it has been conventional in nonrelativistic approximations to ignore the latter. Then $\hat{\rho}(q)$ is given by the spatial Fourier transform of the local nucleon charge density operator:

$$\hat{\rho}(q) \equiv G_{E,p}(Q^2) \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}(\mathbf{x}), \quad (1.4)$$

where $\hat{\psi}(\mathbf{x})$ is the local (nonrelativistic) nucleon field-operator, $\hat{Q} = (1 + \tau_3)/2$ is the charge operator in isospin space which projects out protons, and $G_{E,p}(q^2)$ is the proton charge form factor. (In a purely nonrelativistic treatment of the target, $G_{E,p}(Q^2)$ would be replaced by the Fourier transform of the proton charge density $F_p(\mathbf{q}^2)$.) Charge effects of neutrons are small and are usually neglected.

To proceed to a sum rule, we first define the nonrelativistic Coulomb sum function for inelastic scattering at fixed three-momentum transfer \mathbf{q} :

$$\begin{aligned} S_{NR}(\mathbf{q}) &\equiv \int_{\omega_{el}^+}^{\infty} d\omega \frac{W_C(\omega, \mathbf{q})}{G_{E,p}^2(Q^2)} \\ &= \int d^3x d^3x' e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \sum_{f \neq i} \langle i | \hat{\psi}^\dagger(\mathbf{x}') \hat{Q}^\dagger \hat{\psi}(\mathbf{x}') | f \rangle \langle f | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle, \end{aligned} \quad (1.5)$$

where the integration is over all energies above the elastic peak at $\omega = \omega_{el}$. Expression (1.5) then becomes a sum rule by subtracting the elastic term $f = i$ explicitly and using the closure relation

$$\sum_f |f\rangle \langle f| = 1, \quad (1.6)$$

on the resulting sum over nuclear states $|f\rangle$. Equation (1.5) is then in the form of a Fourier transform of the expectation value of four nucleon field operators. To separate out the two-

¹For notational simplicity, we assume a nondegenerate ground state $|i\rangle$. The results can easily be generalized to unpolarized targets with $J \neq 0$.

body correlation part, we put the operators in normal order (creation operators to the left) by anticommutation, using

$$\{\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')\} = \delta(\mathbf{x} - \mathbf{x}'). \quad (1.7)$$

The sum rule then becomes

$$\begin{aligned} S_{NR}(\mathbf{q}) &= \int d^3x \langle i | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle \\ &+ \int d^3x d^3x' e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \left\{ \langle i | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}^\dagger(\mathbf{x}') \hat{\psi}(\mathbf{x}') \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle \right. \\ &\quad \left. - \langle i | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle \langle i | \hat{\psi}^\dagger(\mathbf{x}') \hat{Q} \hat{\psi}(\mathbf{x}') | i \rangle \right\}. \end{aligned} \quad (1.8)$$

The first term is simply the total nuclear charge Z , and the second term can be written in terms of a correlation function:

$$\begin{aligned} C_{NR}(\mathbf{q}) &= \int d^3x d^3x' e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \left\{ \langle i | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}^\dagger(\mathbf{x}') \hat{\psi}(\mathbf{x}') \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle \right. \\ &\quad \left. - \left(\frac{Z-1}{Z} \right) \langle i | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle \langle i | \hat{\psi}^\dagger(\mathbf{x}') \hat{Q} \hat{\psi}(\mathbf{x}') | i \rangle \right\}, \end{aligned} \quad (1.9)$$

and the elastic form factor of the nuclear target:

$$F_{el}(\mathbf{q}) = G_{E,p}(Q_{el}^2) \int d^3x e^{i\mathbf{q} \cdot \mathbf{x}} \langle i | \hat{\psi}^\dagger(\mathbf{x}) \hat{Q} \hat{\psi}(\mathbf{x}) | i \rangle, \quad (1.10)$$

where $Q_{el}^2 \equiv \mathbf{q}^2 - \omega_{el}^2$. Combining (1.8)–(1.10), the NRCSR takes the compact form

$$S_{NR}(\mathbf{q}) = Z + C_{NR}(\mathbf{q}) - \frac{1}{Z} \frac{|F_{el}(\mathbf{q})|^2}{G_{E,p}^2(Q_{el}^2)}. \quad (1.11)$$

The function $C_{NR}(\mathbf{q})$ is simply the Fourier transform of the spatial two-proton correlation function. It has been defined so that $C_{NR}(\mathbf{q}) = 0$ for a system of Z uncorrelated protons. Since $F_{el}(\mathbf{q})$ is measured by elastic (e, e') scattering, the NRCSR (1.11) provides a direct measure of proton-proton correlations. Since both $F_{el}(\mathbf{q})$ and $C_{NR}(\mathbf{q})$ are expected to vanish in the limit of large $|\mathbf{q}|$, the approach of $S_{NR}(\mathbf{q}) \rightarrow Z$ with increasing $|\mathbf{q}|$ is a test of the adequacy of the theoretical assumptions.

Experimental studies of (e, e') carried out in the momentum transfer region up to $|\mathbf{q}| = 550$ MeV/c over the last decade have left serious questions about the status of the NRCSR.

For light targets: $A = 2-4$, the measured Coulomb sum $S_{NR}(\mathbf{q}) \rightarrow Z$ for the largest momentum values reached.[2] For heavier targets, however, $S_{NR} < Z$ by 20-30% compared to theoretical predictions based on independent particle models,[3–8] with larger suppressions occurring for larger nuclei. This is despite the fact that Pauli correlations are expected to vanish for $|\mathbf{q}| > 2p_F \simeq 550$ MeV/c. We shall not discuss all the possible effects which could be involved here, and which have been reviewed elsewhere.[9, 10] However, one problem which is always present in the application of the NRCSR to data is whether the sum defined in (1.5) is in fact saturated by the experimental data: not only is there an absolute relativistic limit $\omega < |\mathbf{q}|$ due to the spacelike nature of virtual photons in electron scattering, but other experimental difficulties prevent one from reaching even that limit in practice. We expect that extending experiments to higher energies and momenta, such as will be done at CEBAF, may improve this situation.

The main problem dealt with in this paper is the extension of the Coulomb sum rule to include the relativity of nucleons in inelastic scattering. This becomes unavoidable for momentum transfers $|\mathbf{q}| \sim M$, the nucleon mass, even if the nuclear target is essentially non-relativistic. This follows because even if the nucleons in a typical nucleus move at essentially nonrelativistic velocities, they become highly relativistic after the absorption of a virtual photon carrying large three-momentum. Thus to obtain an adequate description, one must describe nucleons with relativistic wavefunctions, e.g., solutions to the Dirac equation. This in turn introduces the difficulty of treating antinucleon degrees of freedom, which we deal with approximately. Other issues also arise at higher energies, including the composite nature of nucleons themselves. This includes both nucleon electromagnetic form factors, which we include, and inelastic excitations of nucleons, which we do not. We also do not consider the effects of exchange currents among nucleons, e.g., as carried by charged mesons. These have been studied by Schiavilla *et al.*, [11] for example. For the purpose of this paper, the nucleus consists of interacting relativistic nucleons, considered to be elementary particles with form factors.

A number of authors have considered relativistic extensions of the NRCSR with the approximations just mentioned. Walecka[12] made the most direct extension by integrating the relativistic Coulomb response function over all energies to allow the use of closure. This results in a sum rule which is nearly identical in form to the NRCSR. Matsui[13] showed, however, that such a sum rule would never be saturated by electron scattering, for which $\omega < |\mathbf{q}|$. He showed that, in the Fermi gas model, excitations of the Fermi sea (N scattering) are found entirely in the spacelike response, while excitations of the Dirac sea ($N\bar{N}$ pair-production) are found entirely in the timelike response. Unlike the NRCSR, for a system of

Z Dirac protons the spacelike relativistic Coulomb sum tends to $Z/2$ in the high- $|\mathbf{q}|$ limit as a result of this separation. This work initiated interest in spacelike Coulomb sum rules based on energy transfers of $\omega < |\mathbf{q}|$ only. Several authors have studied the case of a Fermi gas target.[13–16] We will see that the exclusion of timelike contributions from the Coulomb sum introduces kinematic factors related to relativistic nucleon recoil which must be accounted for in the formulation of *sum rules* for large $|\mathbf{q}|$.

DeForest[17] recognized the issue of relativistic nucleon recoil and proposed a correction in the form of a modified electric form factor: $\bar{G}_E(Q^2) \equiv G_E(Q^2)\sqrt{(1+\tau)/(1+2\tau)}$, where $\tau \equiv Q^2/4M^2$, which would be factored from the response function in the definition of the relativistic Coulomb sum. He conjectured that the resulting sum rule should have an interpretation similar to the NRCSR, but did not show this. DeForest’s prescription has been used in the analysis of recent data.[18] However, we will show (in Sec. 5) that a result similar to his can be derived (RCSR-I) and does lead to a sum rule which bears close resemblance similar to the NRCSR, but is only accurate for Dirac nucleons, i.e., nucleons without anomalous magnetic moments. Donnelly *et al.*[16] have also proposed a modification to the definition of the spacelike Coulomb sum. By requiring that their modified Coulomb sum tend to Z in the large- $|\mathbf{q}|$ limit, an integral equation for the modified form factor can be derived, then solved via an expansion in moments of the nucleon momentum. In lowest-order, this corresponds to the DeForest prescription evaluated at zero nucleon momentum. However, in higher order, this method introduces an energy dependence into the Coulomb sum function which then is *not* equivalent to a *non-energy-weighted* sum rule of the NRCSR form, from which correlation information can be extracted. We discuss these issues further in the concluding section.

In this paper we derive a spacelike Coulomb sum rule valid for arbitrary three-momentum transfers, in which the relativity of the nucleons is taken into account. We assume that nucleons are the only degrees of freedom contributing to the spacelike sum and ignore, for example, antinucleon degrees of freedom, meson exchange currents and internal excitations of nucleons. In Section 2, we review the formalism for electron scattering from nuclear targets in the first-order Born approximation, including nucleon form factors with an assumption about their off-shell continuation. In Section 3, we consider elastic scattering from a single, free nucleon at rest as the simplest example of relativistic nucleon recoil effects in the constant- \mathbf{q} response; this is shown to become a large effect for $|\mathbf{q}| \sim M$. In Section 4, we develop our “nucleons-only” approximation and derive a relativistic Coulomb sum rule which is exact in its treatment of nucleon recoil and Fermi motion. We give an explicit form for the two-body correlation function. In Section 5, we derive an expansion of the one-body term in

moments of the nucleon momentum. This leads to practical sum rules for the analysis of experimental data. In Section 6, we present numerical results, including a test of our moment expansion in the Fermi gas model. Section 7 contains a discussion and conclusions, including a recommendation for the application of these sum rules to data analysis.

2 Formalism for (e, e') Scattering

We begin with the first-order Born approximation for the inelastic scattering of ultrarelativistic ($|\mathbf{k}| \gg m_e$) electrons from nuclear targets. Initial and final electron four-momenta are denoted by $k^\mu = (E, \mathbf{k})$ and $k'^\mu = (E', \mathbf{k}')$, and initial and final target four-momenta are denoted by $P^\mu = (P^0, \mathbf{P})$ and $P'^\mu = (P'^0, \mathbf{P}')$, respectively. The differential cross-section can be written in factored form:

$$\frac{d^2\sigma}{d\Omega' dE'} = \frac{\alpha^2}{q^4} \frac{|\mathbf{k}'|}{|\mathbf{k}|} L_e^{\mu\nu}(k', k) W_{\mu\nu}(P', P), \quad (2.1)$$

where $q^\mu = (\omega, \mathbf{q}) \equiv k^\mu - k'^\mu = P'^\mu - P^\mu$ is the four-momentum transferred to the target in the laboratory frame by a single virtual photon, and α is the fine-structure constant. The lepton current tensor for an unpolarized electron beam is:

$$L_e^{\mu\nu}(k', k) \equiv \frac{1}{2} \sum_{ss'} [\bar{u}_{s'}(\mathbf{k}') \gamma^\mu u_s(\mathbf{k})] [\bar{u}_{s'}(\mathbf{k}') \gamma^\nu u_s(\mathbf{k})]^*, \quad (2.2)$$

where the fermion spinors $u_s(\mathbf{k})$ are given in (A3).

All of the interesting target physics is contained in the nuclear response function:

$$W_{\mu\nu}(P, q) = \sum_f \langle f | \hat{J}_\mu(q) | i \rangle \langle f | \hat{J}_\nu(q) | i \rangle^* \delta(\omega - E_f + E_i), \quad (2.3)$$

where $|i\rangle$ and $|f\rangle$ denote initial² and final nuclear many-body states, respectively. The tensor $W_{\mu\nu}(P, q)$ may be decomposed into longitudinal and transverse response functions, with respect to the polarization of the virtual photon in a given frame: normally the target rest-frame. (We will not discuss the transverse response further in this paper.) If the z-axis is chosen along the three-momentum transfer \mathbf{q} , i.e., $q^\mu = (\omega, 0, 0, |\mathbf{q}|)$, then the longitudinal polarization vector, chosen to obey $e_L \cdot q = 0$ (Lorentz gauge) and $e_L^2 = 1$, can be written $e_L^\mu = (|\mathbf{q}|, 0, 0, \omega)/\sqrt{Q^2}$. The longitudinal response function is then defined

$$W_L(\omega, \mathbf{q}) \equiv e_L^{\mu*} W_{\mu\nu} e_L^\nu. \quad (2.4)$$

²Here, as in Section 1, we assume a nondegenerate ground state $|i\rangle$.

For theoretical reasons, it will be more convenient to work in the Coulomb gauge, defining the Coulomb response function

$$W_C(\omega, \mathbf{q}) \equiv W_{00}(\omega, \mathbf{q}) = \frac{\mathbf{q}^2}{Q^2} W_L(\omega, \mathbf{q}), \quad (2.5)$$

where the second relation reflects this change of gauge. It is the Coulomb (not longitudinal) response function which appears in (1.1).

In general, the current operator $\hat{J}_\mu(q)$ includes contributions from both charged nucleons and charged mesons in the target ground state, as discussed in the introduction. However, for the present work we ignore contributions to (2.3) from meson currents. This allows us to write the nucleon current operator in the one-body form:

$$\hat{J}_\mu(q) \equiv \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} \bar{\hat{\psi}}(\mathbf{x}) \Gamma_\mu \hat{\psi}(\mathbf{x}), \quad (2.6)$$

where $\hat{\psi}(\mathbf{x})$ is the Dirac nucleon field operator in the Schrödinger picture. In general, the field operators and therefore the current operator (2.6) involve both nucleon and antinucleon degrees of freedom, as well as a sum over isospin projections.

In the relativistic impulse approximation, the field operators in (2.6) refer to free particles. It is then natural to expand these field operators in terms of free Dirac wavefunctions, as in (A2). The γNN vertex operator Γ_μ may then be specified in terms of its matrix elements between Dirac plane-wave spinors: $\bar{u}_{s'}(\mathbf{p}') \Gamma_\mu u_s(\mathbf{p})$, therefore it is conventional to express Γ_μ in terms of Dirac matrices:

$$\Gamma_\mu = F_1 \gamma_\mu + i \frac{\kappa}{2M} F_2 \sigma_{\mu\nu} q^\nu, \quad (2.7)$$

where κ is the nucleon anomalous magnetic moment and M is the nucleon mass. In many-body calculations involving both protons and neutrons, (2.7) includes an implicit sum over proton and neutron isospin projections. In general, the form factors F_1 and F_2 are scalar functions of p , p' and q .

For scattering from free nucleons, the form factors F_1 and F_2 can be shown to be functions only of the scalar variable $Q^2 = \mathbf{q}^2 - \omega^2$. These form factors are obtained from elastic eN scattering data, usually in terms of the more convenient electric and magnetic form factors:

$$\begin{aligned} G_E(Q^2) &\equiv F_1(Q^2) - \kappa \tau F_2(Q^2) \\ G_M(Q^2) &\equiv F_1(Q^2) + \kappa F_2(Q^2) \end{aligned} \quad (2.8)$$

respectively, where $\tau \equiv Q^2/4M^2$. Then F_1 and F_2 are given by

$$\begin{aligned}
F_1(Q^2) &= \frac{G_E(Q^2) + \tau G_M(Q^2)}{1 + \tau} \\
F_2(Q^2) &= \frac{G_M(Q^2) - G_E(Q^2)}{1 + \tau}
\end{aligned} \tag{2.9}$$

with the normalization convention that $F_1^p(0) = F_2^p(0) = F_1^n(0) = 1$ and $F_2^n(0) = 0$.

The continuation of these form factors for interacting nucleons is not unique, requiring dynamical information not contained in the impulse approximation. A common assumption is to use the on-shell values $F_1(Q^2)$ and $F_2(Q^2)$ in constructing the vertex operator (2.7). An equally reasonable assumption, which we adopt in this paper, is to evaluate *only* the electric and magnetic form factors at the off-shell variable Q^2 , i.e., the off-shell F_1 and F_2 are defined:

$$\begin{aligned}
F_1(Q^2, \tilde{Q}^2) &\equiv \frac{G_E(Q^2) + \tilde{\tau} G_M(Q^2)}{1 + \tilde{\tau}} \\
F_2(Q^2, \tilde{Q}^2) &\equiv \frac{G_M(Q^2) - G_E(Q^2)}{1 + \tilde{\tau}}
\end{aligned} \tag{2.10}$$

where $\tilde{\tau} \equiv \tilde{Q}^2/4M^2$ and $\tilde{Q}^2 \equiv \mathbf{q}^2 - (E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}})^2$. Here $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + M^2}$ is the energy of a free nucleon with three-momentum \mathbf{p} .

The crucial feature of the off-shell prescription (2.10) is that the photon energy ω enters Γ_μ in (2.7) only through the form factors $G_E(Q^2)$ and $G_M(Q^2)$. Consider the limit of point nucleons, for which the form factors $F_1(Q^2)$ and $F_2(Q^2)$ in (2.7) become constants: then the plane-wave matrix elements $\bar{u}_{s'}(\mathbf{p}+\mathbf{q})\Gamma_\mu u_s(\mathbf{p})$ depend only on the three-vectors \mathbf{p} and \mathbf{q} . For extended particles, both the F 's and the G 's depend on the scalar Q^2 , and may also depend on another scalar, which we take to be \tilde{Q}^2 . (This choice is not unique, but is sensible given that \mathbf{p} and \mathbf{q} are the only other variables at the γNN vertex in the impulse approximation.) One may assume that *either* the F 's or the G 's, but not both sets, are functions of Q^2 only; we have made the second choice in (2.10). This prescription is necessary to arrive at the particularly simple forms for the relativistic Coulomb sum rule which follows.

We adopt the following parameterizations[19] for the nucleon electric and magnetic form factors appearing in (2.10):

$$G_{E,p}(Q^2) = (1 + Q^2/.71\text{GeV}^2)^{-2} \quad (2.11)$$

$$G_{M,p}(Q^2) = (1 + \kappa_p)G_{E,p}(Q^2) \quad (2.12)$$

$$G_{M,n}(Q^2) = \kappa_n G_{E,p}(Q^2) \quad (2.13)$$

$$G_{E,n}(Q^2) = 0 \quad (2.14)$$

where $\kappa_p = +1.79$ and $\kappa_n = -1.91$ are the proton and neutron anomalous magnetic moments, respectively. Although the neutron electric form factor $G_{E,n}(Q^2)$ is not identically zero in Ref. [19], it is known to be small compared to $G_{E,p}(Q^2)$. Other nonzero forms for $G_{E,n}(Q^2)$ have also been given,[19, 20] and could be adopted for use in the present work. However, we shall make use of the fact that *all* form factors have $G_{E,p}(Q^2)$ as a common factor to simplify the form of the relativistic Coulomb sum rule to follow. We note, however, that this proportionality cannot be exact for nonzero $G_{E,n}(Q^2)$, since $G_{E,n}(0) = 0$.

3 Single-nucleon Coulomb Scattering

In this section, we consider the simplest nuclear target, one free nucleon, in the response function formalism of the previous section. This allows us to illustrate directly some of the features of relativity which also occur in the general nuclear case, and suggest how they may be treated in the formulation of relativistic Coulomb sum rules. In this case, the scattering is elastic on the target nucleon; the differential cross-section in the laboratory frame, which may be obtained from (2.1), is the Rosenbluth formula with form factors (2.8). We are interested only in the Coulomb scattering, which is expressed in terms of the Coulomb response function $W_C(\omega, \mathbf{q}) \equiv W_{00}(\omega, \mathbf{q})$.

For a free nucleon, the Coulomb response function (2.5) may be evaluated from (2.3) and (2.6) by specifying that initial and final nuclear states are simply Dirac plane-wave states of momenta \mathbf{p} and $\mathbf{p}' = \mathbf{p} + \mathbf{q}$, respectively. Using the nucleon field operator of (A2), we find:

$$W_C^N(\mathbf{p}, q) = \frac{L_{00}^N(\mathbf{p}, q)}{4E_{\mathbf{p}+\mathbf{q}}E_{\mathbf{p}}} \delta(\omega - E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}}), \quad (3.1)$$

where the 00-component of the nucleon current tensor is given by

$$L_{00}^N(\mathbf{p}, q) \equiv \frac{1}{2} \sum_{ss'} [\bar{u}_{s'}(\mathbf{p}') \Gamma_0 u_s(\mathbf{p})] [\bar{u}_{s'}(\mathbf{p}') \Gamma_0 u_s(\mathbf{p})]^*, \quad (3.2)$$

and the energy factors in the denominator of (3.1) reflect our convention that the fermion spinors (A3) are normalized to $2E$ particles per unit volume. The delta-function in (3.1) guarantees energy conservation for the free nucleon states: $\omega = E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}}$.

Notice that only nucleon, i.e., not antinucleon, states appear in (3.2). The energy required to excite a free particle of momentum \mathbf{p} into a state with momentum $\mathbf{p}+\mathbf{q}$ satisfies $E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}} < |\mathbf{q}|$, hence the process $N \rightarrow N$ is restricted entirely to the spacelike region $\omega < |\mathbf{q}|$. In contrast, the energy required to excite an $N\bar{N}$ pair from the vacuum satisfies $\omega = E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}} > |\mathbf{q}|$, and hence is restricted entirely to the timelike region $\omega > |\mathbf{q}|$. (Pair-production arising due to scattering from a free nucleon may be considered either an excitation of the nucleon itself or a two-body, i.e., meson exchange, process and is ignored here.) Thus the antinucleon components of the field operators of (A2) do not appear in the (spacelike) response function for a free nucleon. We shall find it advantageous to use this separation of excitation spectra to explicitly remove the contribution of $N\bar{N}$ pairs from the relativistic sum rule. (This issue was first raised by Matsui.[13])

Returning to the Coulomb response function, we first give an explicit form for the 00-component of the nucleon current tensor (3.2). After substituting (2.7) and (2.10) into (3.2), a straightforward trace calculation yields

$$L_{00}^N(\mathbf{p}, q) = \frac{G_E^2(Q^2)}{1 + \tilde{\tau}} (E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}})^2 + \frac{G_M^2(Q^2)}{1 + \tilde{\tau}} [\tilde{\tau} (E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}})^2 - (1 + \tilde{\tau}) \mathbf{q}^2], \quad (3.3)$$

where, of course, for a free nucleon $\tilde{\tau} = \tau$ and $\tilde{Q}^2 = Q^2$. If we put the initial nucleon at rest, evaluation of (3.3) at $\mathbf{p} = 0$ yields

$$L_{00}^N(\mathbf{p}, q)|_{\mathbf{p}=0} = 2M(E_{\mathbf{q}} + M) G_E^2(Q^2), \quad (3.4)$$

where we have used $\tilde{\tau} = (E_{\mathbf{q}} - M)/2M$ at $\mathbf{p} = 0$. Thus stationary nucleons couple *only* through their electric form factor. (A sensible feature of the off-shell prescription (2.10) is that this reduction also occurs for interacting nucleons.) The response function is obtained by substituting (3.4) into (3.1).

The Coulomb sum function is obtained by integrating $W_C(\omega, \mathbf{q})/G_{E,p}^2(Q^2)$ over energy transfer ω , as in (1.5). For electron scattering from a free nucleon, it is natural to restrict the integral to include only spacelike ($\omega < |\mathbf{q}|$) photons, thus excluding $N\bar{N}$ pair production:

$$\int_{\omega_{el}^+}^{|\mathbf{q}|} d\omega \frac{W_C^N(\mathbf{p}, q)|_{\mathbf{p}=0}}{G_E^2(Q^2)} = \frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}} = \frac{M}{E_{\mathbf{q}}} \frac{\mathbf{q}^2}{Q^2} \Big|_{\omega=E_{\mathbf{q}}-M} \quad (3.5)$$

where in the second relation we have made use of the nucleon energy-momentum relation at $\mathbf{p}=0$ to express the result in a form which lends itself easily to interpretation. The first factor $M/E_{\mathbf{q}}$ has its origin in the energy denominator $E_{\mathbf{p}+\mathbf{q}}$ in (3.1), which in turn comes from the relativistic normalization of the fermion spinors (A3). The second factor \mathbf{q}^2/Q^2 arises from our choice to work in Coulomb gauge, as can be seen by comparing with (2.5). Taken together, these factors represent the effect of relativistic recoil (in Coulomb gauge) for a single nucleon. The first relation in (3.5) shows that this recoil effect tends to unity in the nonrelativistic limit $|\mathbf{q}| \ll M$, and tends to the value $1/2$ in the ultrarelativistic limit $|\mathbf{q}| \gg M$. (It is in this sense that Coulomb gauge is a convenient choice, since the analog of (3.5) in Lorentz gauge tends to zero in the limit $|\mathbf{q}| \gg M$.)

Matsui[13] and DoDang *et al.*[14] have previously noticed this limiting value of $1/2$ for the spacelike Coulomb sum, for a Fermi gas of Dirac ($G_E = G_M = 1$) nucleons. The Fermi gas result is easily obtained by inserting (3.3) into (3.1) and integrating over the Fermi sphere $|\mathbf{p}| < p_F$ with an appropriate factor to account for Pauli blocking, as is done in (6.2). It is easy to show that this quantity approaches $Z/2$ (for Z protons) as $|\mathbf{q}| \rightarrow \infty$. That this is a purely kinematic effect of relativistic nucleon recoil is clear from the single-nucleon example (3.5).

The high- $|\mathbf{q}|$ behavior of (3.5) is in contrast to that of the NRCSR (1.11), for which $S_{NR}(\mathbf{q}) \rightarrow Z$ ($Z=1$ here) as $|\mathbf{q}| \rightarrow \infty$, suggesting that in the relativistic limit, the Coulomb sum does *not* directly count the number of charged scatterers, as in the nonrelativistic case. A simple modification of (3.5) would, however, seem to remedy this for the case of a single nucleon: divide both sides by the kinematic recoil factor on the right-hand side, so that the large- $|\mathbf{q}|$ limit is now unity, as would be expected for a sum rule. That this actually corresponds to a sum rule for the many-body case, with a suitable modification for the nucleon momentum distribution (Fermi motion) in the target, is shown in the next two sections.

A final note: For a free nucleon, the delta-function of (3.1) appears in all components of $W_{\mu\nu}$. To obtain the Rosenbluth formula, one must integrate (2.1) over dE' (or $d\omega$) at fixed electron scattering angle θ , rather than at fixed \mathbf{q} , as in the Coulomb sum (3.5). The constraint that ω and \mathbf{q} are *not* independent variables, but are related by $Q^2 = 4EE' \sin^2(\theta/2)$, then gives rise to the familiar recoil factor:

$$\frac{E'}{E} = \left[1 + \frac{2E}{M} \sin^2\left(\frac{\theta}{2}\right) \right]^{-1} \quad (3.6)$$

which reduces to unity in the nonrelativistic limit $E \ll M$, where E is the electron beam energy. Thus it is well-known that not only form factors, but also nucleon recoil factors,

modify the differential cross-section for electron scattering in the relativistic domain.

4 Relativistic Coulomb Sum Rule for Interacting Nucleons

In this section, we derive a sum rule for a target of interacting nucleons, where the relativity of the nucleons is taken into account. Following the example of a single nucleon in the previous section, we find that if we restrict the sum to spacelike four-momentum transfers, which corresponds to (e, e') scattering experiments, we are led to a “nucleons-only” approximation in which antinucleon degrees of freedom are ignored. The new feature in the sum rule given here is the kinematic correction for relativistic recoil, which depends on the momentum of the nucleons in the target. In the following section, we show how this correction can be calculated in successive approximations in terms of moments of the nucleon momentum.

We begin with the Coulomb response function in the rest-frame of a many-body nuclear system, which is given by $W_{00}(P, q)$ of (2.3), evaluated at $\mathbf{P}=0$:

$$W_C(\omega, \mathbf{q}) = \sum_f |M_{fi}(\omega, \mathbf{q})|^2 \delta(\omega - E_f + E_i), \quad (4.1)$$

where the Coulomb transition matrix element is:

$$M_{fi}(\omega, \mathbf{q}) \equiv \langle f | \hat{J}_0(\omega, \mathbf{q}) | i \rangle = \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} \langle f | \bar{\hat{\psi}}(\mathbf{x}) \Gamma_0 \hat{\psi}(\mathbf{x}) | i \rangle. \quad (4.2)$$

These are the relativistic equivalents of (1.3) and (1.4). The γNN vertex operator Γ_0 is given by (2.7) and includes an implicit sum over isospin projections, and the field operator $\hat{\psi}(\mathbf{x})$ is given in (A2) and includes both nucleon and antinucleon degrees of freedom. As discussed in Section 2, the nucleon form factors are assumed to be given by (2.10) such that the only ω dependence in $\hat{J}_0(q)$ enters through the nucleon electric and magnetic form factors, $G_E(Q^2)$ and $G_M(Q^2)$. However, given the specific functional forms in (2.11)–(2.14), one can instead consider *all* the ω dependence in (4.2) to enter only through the common form factor $G_{E,p}(Q^2)$. Dividing (4.2) by $G_{E,p}(Q^2)$ therefore removes *all* the ω dependence from the nuclear matrix element M_{fi} . Dividing (4.1) by $G_{E,p}^2(Q^2)$ then allows us to generate a *non-energy-weighted* sum rule, as follows.

We consider the Coulomb sum function defined by

$$\Sigma(\mathbf{q}) \equiv \int_{\omega_{el}^+}^{|\mathbf{q}|} d\omega \frac{W_C(\omega, \mathbf{q})}{G_{E,p}^2(Q^2)}, \quad (4.3)$$

where the integration is performed only over energies $\omega < |\mathbf{q}|$, corresponding to electron scattering experiments, as in (3.5). This is unlike the nonrelativistic Coulomb sum (1.5), in which the integration is over all $\omega > \omega_{el}$ in order to invoke closure. Substituting (4.1) into (4.3) and performing the integration, we obtain:

$$\Sigma(\mathbf{q}) = \sum_{f \neq i}^{f: \omega < |\mathbf{q}|} \left| \frac{M_{fi}(\omega, \mathbf{q})}{G_{E,p}(Q^2)} \right|^2, \quad (4.4)$$

where the sum over all final states in (4.1) is now reduced to a sum over only those final states $f \neq i$ which are accessed by spacelike four-momentum transfers. For the case of a free nucleon target at rest, (4.4) reduces to (3.5).

The nucleon field operator (A2), and therefore the transition matrix elements (4.2), involve both nucleon and antinucleon degrees of freedom. However, because electron scattering experiments produce spacelike virtual photons exclusively, the resulting nuclear excitations are restricted to those whose excitation energies satisfy $\omega = E_f - E_i < |\mathbf{q}|$, as indicated in (4.4). We have already seen in (3.5), for the case of a free nucleon, that restricting the energy transfer ω removes the contribution of $N\bar{N}$ pair production from the Coulomb sum, which would enter if $\omega > |\mathbf{q}|$ were included in (4.3). Similarly, for a noninteracting Fermi gas, only excitations involving $a_{\mathbf{p}'s'}^\dagger a_{\mathbf{p}s}$ (N scattering) enter into the spacelike sum. This is because there are no antinucleons present in the renormalized nuclear ground state, thus the terms involving $b_{\mathbf{p}s}^\dagger b_{\mathbf{p}'s'}$ (\bar{N} scattering) and $b_{\mathbf{p}'s'} a_{\mathbf{p}s}$ ($N\bar{N}$ pair-annihilation) vanish identically, and because the terms involving $a_{\mathbf{p}'s'}^\dagger b_{\mathbf{p}s}^\dagger$ ($N\bar{N}$ pair-creation) are restricted entirely to the timelike region, since $\omega = E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}} > |\mathbf{q}|$. Furthermore, the separation of nucleon and antinucleon degrees of freedom is complete in the Fermi gas model: not only are antinucleon scattering and $N\bar{N}$ pair terms completely removed from the spacelike sum, but the $N \rightarrow N$ contribution is guaranteed to lie completely in the spacelike region through the relation $\omega = E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}} < |\mathbf{q}|$. (This feature is unaffected by the inclusion of scalar and vector potentials like those in Quantum Hadrodynamics, for example, as noted by Do Dang *et al.*[14]) Thus, in the Fermi gas model, the spacelike nuclear response (4.4) can be represented by effectively dropping all reference to antinucleon degrees of freedom at the level of the field operator (A2).

For an interacting nuclear system, however, the separation of nucleon and antinucleon degrees of freedom is no longer exact. In general, both N and \bar{N} degrees of freedom can contribute to the spacelike nuclear response, and some of the nucleons-only response $N \rightarrow N$, which is entirely spacelike in the Fermi gas model, can be pushed into the timelike region by two-particle interactions. However, we can assume that true $N\bar{N}$ pair production will still occur mostly for $\omega > |\mathbf{q}|$, and therefore can be neglected in the Coulomb sum (4.4). Although

tests of this assumption are outside the scope of this paper, we note that it is satisfied exactly for infinite nuclear matter in the RPA, i.e., a spacelike photon excites only spacelike excitations even through the summation of an infinite number of RPA ring diagrams, and does not produce $N\bar{N}$ pairs. Antinucleon contributions, i.e., those involving $b_{\mathbf{p}s}^\dagger b_{\mathbf{p}'s'}$ and $b_{\mathbf{p}'s'} a_{\mathbf{p}s}$, enter as relativistic effects in the nuclear target structure. In Hartree calculations for finite nuclei, these terms make only small ($\sim 1\%$) contributions to the relevant (vector) density,[21] and shall also be neglected in this paper. Therefore, we adopt a “nucleons-only” approximation, in which antinucleon degrees of freedom are removed from discussion at the level of the field operator (A2). In effect, we are assuming that the *dominant* relativistic effect comes from single-nucleon recoil at high $|\mathbf{q}|$, and not from relativistic aspects of nuclear structure. Thus the initial and final nuclear states in (4.4) will be treated as interacting many-nucleon states without \bar{N} components.

Substituting the field operators given by (A2) into the transition matrix element (4.2), we have in our “nucleons-only” approximation:

$$M_{fi}(\omega, \mathbf{q}) \simeq G_{E,p}(Q^2) \sum_{\mathbf{p}\sigma} \sum_{ss'} j_{s's,\sigma}(\mathbf{p}, \mathbf{q}) \langle f | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle, \quad (4.5)$$

where the “reduced” spinor matrix element is defined:

$$j_{s's,\sigma}(\mathbf{p}, \mathbf{q}) \equiv \frac{\bar{u}_{s'\sigma}(\mathbf{p}+\mathbf{q})}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}} \frac{\Gamma_0(\mathbf{p}, q)}{G_{E,p}(Q^2)} \frac{u_{s\sigma}(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}}. \quad (4.6)$$

In (4.5) and (4.6), nucleon spin states are represented by Latin indices and nucleon isospin states are represented by Greek indices. The operator $\Gamma_0(\mathbf{p}, q)$ is diagonal in isospin space, therefore the transition matrix element (4.5) simply involves a sum over proton and neutron isospin projections. The term “reduced” in (4.6) refers to the division by $G_{E,p}(Q^2)$, following (4.3). The matrix element (4.6) is then a function only of \mathbf{p} and \mathbf{q} , i.e., not of ω , through our assumption of proportional form factors. Substituting (4.5) into (4.4) leads to:

$$\begin{aligned} \Sigma(\mathbf{q}) = & \sum_{\mathbf{p}\mathbf{p}'} \sum_{rr'} \sum_{ss'} \sum_{\sigma\rho} j_{r'r,\rho}^*(\mathbf{p}', \mathbf{q}) j_{s's,\sigma}(\mathbf{p}, \mathbf{q}) \\ & \left\{ \sum_{f: \omega < |\mathbf{q}|} \langle i | a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} | f \rangle \langle f | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right. \\ & \left. - \langle i | a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} | i \rangle \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right\}, \end{aligned} \quad (4.7)$$

where we have subtracted the elastic term $f=i$ explicitly, in order to obtain a sum over the complete set of nuclear final states. For notational simplicity, we assume a nondegenerate ground state $|i\rangle$. Equation (4.7) is then the most general form for the Coulomb sum function (4.3), given our nucleons-only approximation and our assumptions about nucleon form factors and their off-shell continuation.

In order to obtain a sum rule from (4.7), we must sum over a complete set of final states $|f\rangle$ and use closure. The completeness of such a set necessarily includes excited nuclear states accessed by *both* spacelike ($\omega < |\mathbf{q}|$) and timelike ($\omega > |\mathbf{q}|$) photons, which can be expressed

$$\sum_{f:\omega < |\mathbf{q}|} |f\rangle\langle f| + \sum_{f:\omega > |\mathbf{q}|} |f\rangle\langle f| = 1. \quad (4.8)$$

It is consistent with the nucleons-only approximation imposed in (4.5) to neglect the contribution of timelike states $f : \omega > |\mathbf{q}|$ to the closure of the sum in (4.7), since these states predominantly involve antinucleons. For a uniform Fermi gas in the Hartree-Fock approximation, we have already seen that the separation between N and \bar{N} degrees of freedom guarantees that the spacelike Coulomb sum be completely exhausted in the nucleons-only approximation. Similarly, for an interacting Fermi gas in the RPA, nucleons are excited *only* for $\omega < |\mathbf{q}|$, and $N\bar{N}$ pairs are produced *only* for $\omega > |\mathbf{q}|$, thus the same argument holds. It is possible, however, that restricting the sum in (4.4) to spacelike states $f : \omega < |\mathbf{q}|$ does *not* in general exhaust the nucleons-only excitation spectrum, e.g., finite size effects and two-body interactions beyond the RPA may push some of the nucleons-only response into the timelike region. We are aware of no general argument which applies. We will simply assume that the nucleons-only excitation spectrum *is* saturated by states $f : \omega < |\mathbf{q}|$, so that we can invoke closure in (4.7). Extension of the sum to include nucleonic excitations with $\omega > |\mathbf{q}|$ could be done by theoretical means, as in Schiavilla *et al.*, [22] for example. Applying closure to the first term in (4.7) removes the sum over final states, and the expression in curly brackets becomes:

$$\begin{aligned} \left\{ \dots \right\} &\rightarrow \left\{ \langle i | a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right. \\ &\quad \left. - \langle i | a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} | i \rangle \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right\}. \end{aligned} \quad (4.9)$$

Equation (4.7) is now a *sum rule*, equating the sum (4.3) to the ground state expectation values in (4.9).

To cast the sum rule in a more transparent form, we separate out the one- and two-body parts of the operator in the first term of (4.9), using the momentum-space anticommutation relations for the nucleon creation and destruction operators:

$$\{a_{\mathbf{p}s\sigma}, a_{\mathbf{p}'s'\sigma'}^\dagger\} = \delta_{\mathbf{p}\mathbf{p}'} \delta_{ss'} \delta_{\sigma\sigma'}, \quad (4.10)$$

where the anticommutator of any two creation or destruction operators vanishes. Moving all creation operators to the left in the first term of (4.9), we obtain:

$$\begin{aligned} \left\{ \dots \right\} &\rightarrow \left\{ \langle i | a_{\mathbf{p}s\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \delta_{\mathbf{p}\mathbf{p}'} \delta_{r's'} \delta_{\rho\sigma} \delta_{rs} \right. \\ &+ \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} a_{\mathbf{p}s\sigma} | i \rangle \\ &\left. - \langle i | a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} | i \rangle \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right\}, \end{aligned} \quad (4.11)$$

where we have extracted δ_{rs} from the first matrix element, which is diagonal in spin projection. Inserting (4.11) into (4.7) allows us to separate the sum rule expression into one-body, two-body and elastic amplitudes, in complete analogy to the nonrelativistic expression (1.8). Here, however, we have used momentum operators, rather than local field operators, in order to eliminate the antinucleon components.

We separate the sum rule as follows:

$$\Sigma(\mathbf{q}) \equiv \Sigma^{(1)}(\mathbf{q}) + \Sigma^{(2)}(\mathbf{q}). \quad (4.12)$$

The first term gives the one-body contribution:

$$\Sigma^{(1)}(\mathbf{q}) = 2 \sum_{\mathbf{p}\sigma} n_\sigma(\mathbf{p}) r_\sigma(\mathbf{p}, \mathbf{q}), \quad (4.13)$$

where we have defined the momentum distribution function $n_\sigma(\mathbf{p}) \equiv \langle i | a_{\mathbf{p}s\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle$ for isospin projection σ (which we assume is independent of s), and the relativistic recoil factor

$$\begin{aligned} r_\sigma(\mathbf{p}, \mathbf{q}) &\equiv \frac{1}{2} \sum_{ss'} |j_{s's, \sigma}(\mathbf{p}, \mathbf{q})|^2 \\ &= \frac{1}{G_{E,p}^2(Q^2)} \frac{L_{00}^\sigma(\mathbf{p}, q)}{4E_{\mathbf{p}+\mathbf{q}} E_{\mathbf{p}}}, \end{aligned} \quad (4.14)$$

where $L_{00}^\sigma(\mathbf{p}, q)$ is the nucleon tensor component (3.3). In the nonrelativistic limit $|\mathbf{q}| \ll M$, $r_\sigma(\mathbf{p}, \mathbf{q}) \rightarrow 1$ and $\Sigma^{(1)}(\mathbf{q}) \rightarrow Z$, as in (1.8). For relativistic momenta $|\mathbf{q}| \gg M$, however, we have $r_\sigma(\mathbf{p}, \mathbf{q}) < 1$, which represents the kinematic effect of relativistic nucleon recoil, as in (3.5). If the Fermi momentum in (4.13) could be neglected entirely, i.e., if we were to set $\mathbf{p}=0$ in (4.14), then $\Sigma^{(1)}(\mathbf{q})/Z$ would be given by (3.5). The second term of (4.12) is given by

$$\begin{aligned} \Sigma^{(2)}(\mathbf{q}) = & \sum_{\mathbf{p}\mathbf{p}'} \sum_{ss'} \sum_{rr'} \sum_{\sigma\rho} j_{r'r,\rho}^*(\mathbf{p}', \mathbf{q}) j_{s's,\sigma}(\mathbf{p}, \mathbf{q}) \\ & \left\{ \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} a_{\mathbf{p}s\sigma} | i \rangle \right. \\ & \left. - \langle i | a_{\mathbf{p}'r\rho}^\dagger a_{\mathbf{p}'+\mathbf{q}r'\rho} | i \rangle \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right\}. \end{aligned} \quad (4.15)$$

This expression contains the full two-body probability in momentum space, in analogy with the second term of (1.8), but is weighted by the kinematic factor $j_{r'r,\rho}^*(\mathbf{p}', \mathbf{q}) j_{s's,\sigma}(\mathbf{p}, \mathbf{q})$.

To separate out the correlation part of $\Sigma^{(2)}(\mathbf{q})$ as in (1.9), we first evaluate (4.15) for an uncorrelated target ground state, for which we obtain

$$\Sigma_{un}^{(2)}(\mathbf{q}) = - \sum_{\sigma} \frac{1}{N_{\sigma}} \left| \sum_{\mathbf{p}} \sum_{ss'} j_{s's,\sigma}(\mathbf{p}, \mathbf{q}) \langle i | a_{\mathbf{p}+\mathbf{q}s'\sigma}^\dagger a_{\mathbf{p}s\sigma} | i \rangle \right|^2, \quad (4.16)$$

where $N_{\sigma} = Z, N$ for protons and neutrons, respectively. (In the uncorrelated case, there is no contribution in (4.16) from $\sigma \neq \rho$.) We may then define the correlation function

$$C(\mathbf{q}) \equiv \Sigma^{(2)}(\mathbf{q}) - \Sigma_{un}^{(2)}(\mathbf{q}) \quad (4.17)$$

to give a measure of true two-body correlations, and rewrite (4.12)

$$\Sigma(\mathbf{q}) = \Sigma^{(1)}(\mathbf{q}) + C(\mathbf{q}) + \Sigma_{un}^{(2)}(\mathbf{q}), \quad (4.18)$$

in analogy with (1.11). This result will be referred to as the relativistic Coulomb sum rule (RCSR).

The uncorrelated term $\Sigma_{un}^{(2)}(\mathbf{q})$ is related to the square of the elastic target form factor, with the difference that the proton and neutron contributions are added *incoherently* in (4.16). If, however, the contribution of neutrons to (4.16) can be neglected, then

$$\Sigma_{un}^{(2)}(\mathbf{q}) \simeq - \frac{1}{Z} \frac{|F_{el}(\mathbf{q})|^2}{G_{E,p}^2(Q_{el}^2)}, \quad (4.19)$$

completing the analogy of (4.18) to (1.11). For example, if the contribution of the magnetic ($G_M(Q^2)$) terms is zero, as discussed by Friar[23] for a spin-saturated target, then with $G_{E,n}=0$, as assumed in (2.14), (4.19) is exact. For $G_{E,n}\neq 0$, (4.19) can be suitably modified under reasonable assumptions, e.g., similar neutron and proton distributions in (4.16).

The RCSR given here is exact in its treatment of nucleon Fermi motion and in principle allows the identification of nucleon-nucleon correlations to arbitrarily high $|\mathbf{q}|$, given our assumptions about antinucleon and meson degrees of freedom, nucleon excitations, and off-shell continuation of nucleon form factors. From (4.18), the extraction of the correlation function $C(\mathbf{q})$ from the calculated Coulomb sum (4.3) requires the removal of the nuclear elastic form factor, as just discussed, as well as a reliable evaluation of the one-body term (4.13). For the latter, this form of the RCSR presupposes that one has in hand the nucleon momentum distribution $n_\sigma(\mathbf{p})$.

5 Expansion in Moments of Nucleon Momentum

The momentum distribution $n_\sigma(\mathbf{p})$ is not generally available directly from experimental data. In order to use the RCSR as given in the previous section, therefore, one would have to rely on a theoretical model to calculate the one-body term $\Sigma(\mathbf{q})$. An alternative approach, which we develop in this section, is to expand the one-body term (4.13) in moments of the nucleon momentum, i.e., in averages of powers of the nucleon momentum, weighted by the distribution $n_\sigma(\mathbf{p})$. The one-body term (4.13) is then replaced by a sum in terms of these moments, the first few of which may be known. This procedure leads to a series of approximate sum rules, each of which depends on higher momentum moments. In each case, it is possible to identify appropriate modifications to the definition of the relativistic Coulomb sum (4.3) and arrive at a series of modified sum rules, which share certain features with the NRCSR. Of course, such a procedure is useful *only* if the moment expansion converges rapidly, and if an adequate number of moments can be obtained reliably. In the next section, we will demonstrate that this expansion indeed converges quickly for a uniform Fermi gas, and argue in the following discussion that this is likely to be the case in general.

5.1 Sum Rule I

The lowest-order momentum moment expansion is obtained by setting $\mathbf{p}=0$ in (4.14):

$$r_\sigma(\mathbf{p}, \mathbf{q}) \simeq r_\sigma(\mathbf{0}, \mathbf{q}) = \frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}} \delta_{\sigma p}, \quad (5.1)$$

which is just the factor that appeared in (3.5) for a single proton at rest. This is equivalent to approximating the reduced matrix element (4.6) by its value at $\mathbf{p}=0$:

$$j_{s's,\sigma}(\mathbf{p}, \mathbf{q}) \simeq j_{s's,\sigma}(\mathbf{0}, \mathbf{q}) = \sqrt{\frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}}} \delta_{ss'} \delta_{\sigma p}. \quad (5.2)$$

Since we evaluate at $\mathbf{p}=0$, (5.1) and (5.2) do not involve magnetic terms. The absence of neutron electric contributions is a result of our approximation (2.14).

Inserting (5.1) into (4.13), the one-body term becomes

$$\Sigma^{(1)}(\mathbf{q}) \simeq Z \left[\frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}} \right]. \quad (5.3)$$

Similarly, inserting (5.2) into (4.15) and (4.16), the two-body correlation term (4.17) becomes

$$\begin{aligned} C(\mathbf{q}) \simeq & \left[\frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}} \right] \sum_{\mathbf{p}\mathbf{p}'} \sum_{ss'} \left\{ \langle i | a_{\mathbf{p}+\mathbf{q}s}^\dagger a_{\mathbf{p}'s'}^\dagger a_{\mathbf{p}'+\mathbf{q}s'} a_{\mathbf{p}s} | i \rangle \right. \\ & \left. - \left(\frac{Z-1}{Z} \right) \langle i | a_{\mathbf{p}'s'}^\dagger a_{\mathbf{p}'+\mathbf{q}s'} | i \rangle \langle i | a_{\mathbf{p}+\mathbf{q}s}^\dagger a_{\mathbf{p}s} | i \rangle \right\}, \end{aligned} \quad (5.4)$$

where the proton isospin label has been suppressed for clarity.

Since both $\Sigma^{(1)}(\mathbf{q})$ and $C(\mathbf{q})$ now appear with the *same* overall kinematic factor, i.e., that which appeared in (3.5), which is a function only of \mathbf{q} , it is possible to modify the definition of the Coulomb sum (4.3) and obtain a more conventional form for the sum rule. We define the modified Coulomb sum function:

$$S_I(\mathbf{q}) \equiv \frac{\Sigma(\mathbf{q})}{r_I(\mathbf{q})} = \frac{1}{r_I(\mathbf{q})} \int_{\omega_{el}^+}^{|\mathbf{q}|} d\omega \frac{W_C(\omega, \mathbf{q})}{G_{E,p}^2(Q^2)}, \quad (5.5)$$

where the lowest-order recoil correction factor is

$$r_I(\mathbf{q}) \equiv \frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}}, \quad (5.6)$$

and $W_C(\omega, \mathbf{q})$ is to be extracted from experimental data, using (1.1). We then find the *approximate* sum rule:

$$S_I(\mathbf{q}) \simeq Z + \tilde{C}_I(\mathbf{q}) - \frac{1}{r_I(\mathbf{q})} \left[\frac{1}{Z} \frac{|F_{el}(\mathbf{q})|^2}{G_{E,p}^2(Q_{el}^2)} \right]. \quad (5.7)$$

This result would be exact for a target of free, stationary protons. The two-proton correlation function in (5.7) is given by

$$\begin{aligned}\tilde{C}_I(\mathbf{q}) = & \sum_{\mathbf{p}\mathbf{p}'} \sum_{ss'} \left\{ \langle i | a_{\mathbf{p}+\mathbf{q}s}^\dagger a_{\mathbf{p}'s'}^\dagger a_{\mathbf{p}'+\mathbf{q}s'} a_{\mathbf{p}s} | i \rangle \right. \\ & \left. - \left(\frac{Z-1}{Z} \right) \langle i | a_{\mathbf{p}'s'}^\dagger a_{\mathbf{p}'+\mathbf{q}s'} | i \rangle \langle i | a_{\mathbf{p}+\mathbf{q}s}^\dagger a_{\mathbf{p}s} | i \rangle \right\},\end{aligned}\quad (5.8)$$

in close analogy with (1.9). Expression (5.7) will be referred to as RCSR-I.

Like the NRCSR given in (1.11), RCSR-I involves a one-body term Z which simply counts the total number of charged scatterers, and a two-body term which contains information on nucleon-nucleon correlations. The function $\tilde{C}_I(\mathbf{q})$, however, is *not* the Fourier transform of the usual spatial correlation function, as defined in terms of local nucleon field operators, i.e., simply inserting the Dirac nucleon field operators of (A2) into (1.9) will *not* yield (5.8), because of the antinucleon degrees of freedom which are eliminated from (5.8), and the kinematic factors related to relativistic nucleon recoil which result from that elimination. However, if the *nonrelativistic* result (1.9) is expressed in momentum space, it is exactly of the form (5.8). Thus this version of the RCSR, although based on a rather strong assumption about Fermi motion in the target, is closest in form to the familiar NRCSR.

5.2 Sum Rule II

The procedure which led to RCSR-I can be extended to include higher moments of the nucleon momentum. In this subsection, we keep terms only through second order. We first expand the relativistic recoil factor:

$$r_\sigma(\mathbf{p}, \mathbf{q}) = r_0^\sigma + r_1^\sigma (\mathbf{p} \cdot \mathbf{q}) + r_2^\sigma \mathbf{p}^2 + r_3^\sigma (\mathbf{p} \cdot \mathbf{q})^2 + \mathcal{O}(\mathbf{p}^3), \quad (5.9)$$

for isospin projection σ . Since the recoil factor (4.14) is a function only of \mathbf{p} and \mathbf{q} , i.e., not ω , the expansion coefficients r_i^σ are functions only of \mathbf{q} . Explicit forms for the r_i^σ are given in Appendix B, where we find $r_0^n = r_1^n = 0$ as a consequence of approximation (2.14).

Substituting (5.9) into the one-body term (4.13) leads to

$$\Sigma_C^{(1)}(\mathbf{q}) \simeq 2 \sum_{\mathbf{p}} \left\{ n_p(\mathbf{p}) \left[r_0^p + \left(r_2^p + \frac{\mathbf{q}^2}{3} r_3^p \right) \mathbf{p}^2 \right] + n_n(\mathbf{p}) \left[\left(r_2^n + \frac{\mathbf{q}^2}{3} r_3^n \right) \mathbf{p}^2 \right] \right\}, \quad (5.10)$$

where we have used the spherical symmetry of $n_\sigma(\mathbf{p})$ to eliminate the r_1^σ term and integrate the angles: $(\mathbf{p} \cdot \mathbf{q})^2 \rightarrow \frac{1}{3} \mathbf{p}^2 \mathbf{q}^2$. We can now identify the average squared nucleon momentum, i.e., the second momentum moment:

$$\langle \mathbf{p}^2 \rangle_\sigma \equiv \frac{1}{N_\sigma} \left[2 \sum_{\mathbf{p}} n_\sigma(\mathbf{p}) \mathbf{p}^2 \right], \quad (5.11)$$

where $N_\sigma = Z, N$ for protons and neutrons, respectively. The one-body term (5.10) can then be expressed

$$\Sigma_C^{(1)}(\mathbf{q}) \simeq Z \left[r_0^p + \left(r_2^p + \frac{\mathbf{q}^2}{3} r_3^p \right) \langle \mathbf{p}^2 \rangle_p \right] + N \left[\left(r_2^n + \frac{\mathbf{q}^2}{3} r_3^n \right) \langle \mathbf{p}^2 \rangle_n \right]. \quad (5.12)$$

We thus have an approximate expression for the one-body term (4.13) which depends only on the two parameters $\langle \mathbf{p}^2 \rangle_\sigma$, and reduces to (5.3) if $\langle \mathbf{p}^2 \rangle_p = \langle \mathbf{p}^2 \rangle_n = 0$.

We can now follow a procedure similar to that of (5.5), by defining the modified Coulomb sum function:

$$S_{II}(\mathbf{q}) \equiv \frac{\Sigma(\mathbf{q})}{r_{II}(\mathbf{q})} = \frac{1}{r_{II}(\mathbf{q})} \int_{\omega_{el}^+}^{|\mathbf{q}|} d\omega \frac{W_C(\omega, \mathbf{q})}{G_{E,p}^2(Q^2)}, \quad (5.13)$$

where the second-order relativistic recoil factor is

$$r_{II}(\mathbf{q}) \equiv \left[r_0^p + \left(r_2^p + \frac{\mathbf{q}^2}{3} r_3^p \right) \langle \mathbf{p}^2 \rangle_p + \frac{N}{Z} \left(r_2^n + \frac{\mathbf{q}^2}{3} r_3^n \right) \langle \mathbf{p}^2 \rangle_n \right]. \quad (5.14)$$

We thus find the second-order *approximate* sum rule:

$$S_{II}(\mathbf{q}) \simeq Z + \tilde{C}_{II}(\mathbf{q}) - \frac{1}{r_{II}(\mathbf{q})} \left[\frac{1}{Z} \frac{|F_{el}(\mathbf{q})|^2}{G_{E,p}^2(Q_{el}^2)} \right], \quad (5.15)$$

in analogy with (5.7). The second-order nucleon-nucleon correlation function is given by

$$\tilde{C}_{II}(\mathbf{q}) = \frac{C(\mathbf{q})}{r_{II}(\mathbf{q})}, \quad (5.16)$$

where $C(\mathbf{q})$ is defined in (4.17). Expression (5.15) will be referred to as RCSR-II. Explicit expressions for the two-body term $\Sigma(\mathbf{q})$ and the correlation function $\tilde{C}(\mathbf{q})$ could be given to the same order of approximation made in (5.9), but would require the expansion of $j_{s's,\sigma}(\mathbf{p}, \mathbf{q})$ to second-order in \mathbf{p} . Since we shall not use these explicit forms, we do not pursue this here.

Although RCSR-II enjoys a less direct correspondence with the NRCSR than does RCSR-I, it is more accurate in its treatment of Fermi motion. More importantly, as we will see in the next section, the higher-order terms which are included in RCSR-II introduce effects from proton and neutron anomalous magnetic moments which are appreciable.

6 Numerical Results and Test in Fermi Gas Model

In this Section, we present numerical results for our relativistic Coulomb sum rules. We first consider the recoil correction factors $r_I(\mathbf{q})$ and $r_{II}(\mathbf{q})$, which appear in RCSR-I and RCSR-II, respectively, and find that terms $\sim \langle \mathbf{p}^2 \rangle_\sigma$ in our momentum expansion bring in anomalous magnetic moment effects which are appreciable. We then illustrate in the Fermi gas model the accuracy with which RCSR-I and RCSR-II allow the evaluation of the one-body term.

Ideally, one would like to extract $\langle \mathbf{p}^2 \rangle_\sigma$ directly from experimental data. Since that is not always possible, however, one may have to rely upon estimates based on theoretical models. The simplest such model is a uniform Fermi gas, where $n_\sigma(\mathbf{p}) = \theta(p_F^\sigma - |\mathbf{p}|)$. In that case, we have:

$$\langle \mathbf{p}^2 \rangle_\sigma = \frac{3}{5} p_F^\sigma{}^2, \quad (6.1)$$

where p_F^σ is the Fermi momentum for isospin projection σ . In this example, we assume $Z = N$ and take $p_F^\sigma = p_F = 1.42 \text{ fm}^{-1} = .28 \text{ GeV}/c$ to match the average density in the interior of finite nuclei. This gives $\langle \mathbf{p}^2 \rangle_\sigma = .047 \text{ GeV}^2/c^2$. For $Z \neq N$, it is reasonable to assume that the Fermi momenta for protons and neutrons scale according to $p_F^\sigma = p_F (2N_\sigma/A)^{1/3}$.

Figure 1 shows the recoil correction factors $r_I(\mathbf{q})$ and $r_{II}(\mathbf{q})$, given in (5.6) and (5.14), as functions of the three-momentum transfer \mathbf{q} . The dotted curve is $r_I(\mathbf{q})$, and is indistinguishable (on the scale of this plot) from the factor given by DeForest: $\bar{G}_{E,p}^2/G_{E,p}^2 = (1+\tau)/(1+2\tau)$, suitably averaged over angle. The dot-dashed curve is $r_{II}(\mathbf{q})$ with $\kappa_p = \kappa_n = 0$, i.e., for Dirac nucleons. Both curves tend to the value $1/2$ as $|\mathbf{q}| \rightarrow \infty$. Their difference is a measure of the importance of second-order terms in the moment expansion when anomalous magnetic moments are ignored, and is at most $\sim 1/2\%$ anywhere in the range $0 < |\mathbf{q}| < \infty$. The dashed curve is $r_{II}(\mathbf{q})$ with $\kappa_p = 1.79$ and $\kappa_n = 0$, and shows that the proton anomalous magnetic moment leads to a noticeable enhancement in the recoil factor: 5% at $|\mathbf{q}| = 1 \text{ GeV}$, and reaching 24% as $|\mathbf{q}| \rightarrow \infty$, when compared to $r_I(\mathbf{q})$. The solid curve is $r_{II}(\mathbf{q})$ with $\kappa_p = 1.79$ and $\kappa_n = -1.91$, and shows a similar enhancement due to the neutron anomalous magnetic moment when $N = Z$: 2% at $|\mathbf{q}| = 1 \text{ GeV}$, and reaching 13% as $|\mathbf{q}| \rightarrow \infty$, when compared to $r_I(\mathbf{q})$.

It is clear from Figure 1 that $r_{II}(\mathbf{q})$ is significantly different from $r_I(\mathbf{q})$ *only* when anomalous magnetic moments are included. This can be understood by considering the coefficient of $\langle \mathbf{p}^2 \rangle_p$ in (5.14), i.e., the quantity $r_2^p + (\mathbf{q}^2/3)r_3^p$, where r_2^p and r_3^p are given by (B3) and (B4), respectively. We find that the coefficients of $(1+\kappa_p)^0$ and $(1+\kappa_p)^2$ in that quantity are of roughly the same magnitude and have opposite sign. Thus the second-order pro-

ton contribution to (5.14) is nearly zero when $\kappa_p = 0$, and $r_{II}(\mathbf{q}) \simeq r_I(\mathbf{q})$. However, when $\kappa_p = 1.79$, we have $(1 + \kappa_p)^2 \sim 8$. In that case, the coefficients of $(1 + \kappa_p)^0$ and $(1 + \kappa_p)^2$ no longer cancel, and $r_{II}(\mathbf{q})$ is significantly enhanced. Similarly, for neutrons, both r_2^n and r_3^n are positive and proportional to κ_n^2 . Thus when $\kappa_n = -1.91$, we have $\kappa_n^2 \sim 4$ and $r_{II}(\mathbf{q})$ is further enhanced. From the specific functional form of $L_{00}^p(\mathbf{p}, q)$, as given in (3.4), we note that terms in (5.9) which are of higher order in \mathbf{p}^2 will *not* involve higher powers of the anomalous terms $(1 + \kappa_p)^2$ and κ_n^2 . Therefore, we expect that the moment expansion will converge *if* the moments themselves converge.

We now present a numerical test of our moment expansion in the simple system which includes Fermi motion: a uniform Fermi gas. The spacelike Coulomb response function for this system has been studied by several authors.[13–16, 24] Substituting (3.3) into (3.1), and integrating over the Fermi sphere $|\mathbf{p}| < p_F$ with an appropriate factor to account for Pauli blocking, we obtain:

$$W_C(\omega, \mathbf{q}) = 2 \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} n_{\sigma}(\mathbf{p}) [1 - n_{\sigma}(\mathbf{p} + \mathbf{q})] \frac{L_{00}^{\sigma}(\mathbf{p}, \mathbf{q})}{4E_{\mathbf{p}+\mathbf{q}}E_{\mathbf{p}}} \delta(\omega - E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}}), \quad (6.2)$$

where the occupancy function $n_{\sigma}(\mathbf{p}) = \theta(p_F - |\mathbf{p}|)$. Inserting (6.2) into (4.3) gives the relativistic Coulomb sum:

$$\Sigma(\mathbf{q}) = 2 \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} n_{\sigma}(\mathbf{p}) [1 - n_{\sigma}(\mathbf{p} + \mathbf{q})] r_{\sigma}(\mathbf{p}, \mathbf{q}), \quad (6.3)$$

where we have used definition (4.14). Comparison with (4.13) makes the identification of one- and two-body contributions trivial for this model. Furthermore, the two-body term involves only Pauli correlations, which vanish for $|\mathbf{q}| > 2p_F$, and $F_{el}(\mathbf{q}) = 0$ for $|\mathbf{q}| > 0$. We now demonstrate in several cases the accuracy to which $\Sigma(\mathbf{q})$ of (6.3) is represented by RCSR-I and RCSR-II. We take (6.3) to represent the experimental Coulomb sum obtained from (4.3) and (1.1), and divide out the recoil factors $r_I(\mathbf{q})$ and $r_{II}(\mathbf{q})$, as in (5.5) and (5.13).

Figure 2 shows several versions of the Coulomb sum (divided by Z) for a free Dirac proton gas, i.e., $\kappa_p = \kappa_n = 0$, versus three-momentum \mathbf{q} in units of the Fermi momentum p_F . The dot-dashed curve is the unmodified Coulomb sum $\Sigma(\mathbf{q})/Z$, given by (6.3); this is exactly the result given by Matsui,[13] and tends to the value $1/2$ as $|\mathbf{q}| \rightarrow \infty$. The dashed curve is $S_I(\mathbf{q})/Z$, as defined in (5.5), and the solid curve is $S_{II}(\mathbf{q})/Z$, as defined in (5.13). Here, as in Figure 1, the result given by DeForest is indistinguishable on the scale of this plot from RCSR-I. Both the dashed (RCSR-I) and solid (RCSR-II) curves approach unity quickly as $|\mathbf{q}| \rightarrow \infty$, although RCSR-I overshoots slightly ($\sim 1/10\%$) in the 1 GeV region. The

convergence to unity as $|\mathbf{q}| \rightarrow \infty$ is guaranteed in the Dirac case, since both the unmodified Coulomb sum $\Sigma(\mathbf{q})/Z$ and the recoil factors $r_I(\mathbf{q})$ and $r_{II}(\mathbf{q})$ approach the value $1/2$, for $\kappa_p = \kappa_n = 0$.

Figure 3 shows the same quantities for a free proton gas, with $\kappa_p = 1.79$. In this case, $\Sigma(\mathbf{q})/Z \rightarrow .62$ as $|\mathbf{q}| \rightarrow \infty$, due to the proton anomalous magnetic moment. In contrast to the Dirac case, RCSR-I *fails* to level off at $|\mathbf{q}| = 2p_F$, but rather $S_I(\mathbf{q})/Z \rightarrow 1.23$ as $|\mathbf{q}| \rightarrow \infty$. However, since $r_{II}(\mathbf{q})$ accounts for the anomalous moment, RCSR-II levels off immediately at $|\mathbf{q}| = 2p_F$ and $S_{II}(\mathbf{q})/Z \rightarrow .99$ as $|\mathbf{q}| \rightarrow \infty$. The difference from unity is a measure of the importance of higher-order terms, e.g., $\langle \mathbf{p}^4 \rangle_\sigma$, which have been neglected in our expansion. Figure 4 shows the same results for a symmetric ($N = Z$) Fermi gas. In this case, neutron terms further enhance the unmodified sum: $\Sigma(\mathbf{q})/Z \rightarrow .68$ and $S_I(\mathbf{q})/Z \rightarrow 1.35$ as $|\mathbf{q}| \rightarrow \infty$. However, when the neutron anomalous moment is incorporated into the recoil factor $r_{II}(\mathbf{q})$, we again have $S_{II}(\mathbf{q})/Z \rightarrow .99$ as $|\mathbf{q}| \rightarrow \infty$.

We have shown that it is possible to estimate reliably the one-body term in the Coulomb sum if one accounts correctly for nucleon recoil. Due to the simplicity of the Fermi gas model, however, the numerical study given here does *not* test the validity of our nucleons-only approximation, since the separation of N and \bar{N} degrees of freedom is exact in this case, nor does it test our assumptions about the off-shell continuation of nucleon form factors, since $\tau = \bar{\tau}$ in this case. This study does, however, give an indication of the importance of anomalous magnetic moment effects, and how accurate these approximate sum rules *can* be for the extraction of NN correlation information from electron scattering data, when the momentum distribution of the nuclear target is like that of a Fermi gas.

7 Discussion and Conclusions

The main result of this paper is the derivation of a Coulomb sum rule applicable to the analysis of inelastic electron scattering experiments on nuclei at high three-momentum transfers. The restriction to spacelike four-momenta ($\omega < |\mathbf{q}|$), appropriate for electron scattering, approximately eliminates $N\bar{N}$ pair-production from the response function. We then ignore the small contributions which arise due to antinucleons in the target ground state; this is our “nucleons-only” approximation. The sum rule is given in three forms.

The most general form of the relativistic Coulomb sum rule (RCSR) appears in (4.18). As for the well-known non-relativistic case (NRCSR), the RCSR can be decomposed into a one-body term ($\Sigma^{(1)}(\mathbf{q})$) and a two-body correlation term ($C(\mathbf{q})$) after removing the elastic term. The correlation term is of considerable interest, but is expected to be dominated by

the one-body term for all but the smallest three-momentum transfers. In the NRCSR, the one-body term is simply Z , the number of protons. In the RCSR, however, this term is a function of \mathbf{q} and includes a kinematic factor $r_\sigma(\mathbf{p}, \mathbf{q})$, as shown in (4.13). This factor can be interpreted as the relativistic effect of nucleon recoil, and becomes appreciable for $|\mathbf{q}| \sim M$. In general, the evaluation of (4.13) requires knowledge of the full momentum distribution function $n_\sigma(\mathbf{p})$, which is not accurately known from experiments. This then limits the direct applicability of the RCSR, and in particular, the extraction of correlation information from the analysis.

We therefore develop an expansion of the one-body term in moments of the nucleon momentum. This leads to approximate sum rules, which require only *partial* knowledge of the nucleon momentum distribution $n_\sigma(\mathbf{p})$, i.e., the first few moments, assuming that the moment expansion is rapidly convergent. The simplest case is RCSR-I, given in (5.7), which involves only the zeroth moment. This version of the sum rule requires no information about $n_\sigma(\mathbf{p})$, and therefore has the most direct correspondence with the familiar NRCSR. Although RCSR-I is surprisingly accurate ($\sim 1/10\%$) in approximating the one-body term for a uniform system of Dirac protons, it neglects the contributions from the anomalous magnetic moments, which turn out to be non-negligible ($\sim 23\%$ for $|\mathbf{q}| \gg M$ and $Z = N$). Therefore, we keep the second moment contribution to obtain RCSR-II, given in (5.15), which allows the evaluation of the one-body term to within $\sim 1\%$ for a uniform Fermi gas. This version of the sum rule requires knowledge of only two parameters, namely $\langle \mathbf{p}^2 \rangle_\sigma$ for protons and neutrons, which are approximately known from experimental and theoretical information; we have used the Fermi gas model for our estimate. This study shows that the uncertainty in $\langle \mathbf{p}^2 \rangle_\sigma$ is likely to introduce much less error into the analysis than would ignoring the anomalous magnetic moments altogether. The excellent convergence of the moment expansion, shown here to second-order for a uniform Fermi gas, is not expected to be very different for correlated systems (see, for example, Donnelly *et al.* [16]). Furthermore, the extension of this approach to include higher moments, if necessary, is straightforward. We therefore conclude that RCSR-II, which includes contributions from anomalous magnetic moments, is the most efficient and reliable method for evaluating the sum rule for (e, e') data and separating the two-body information from the dominant one-body term.

We now discuss the relation of our work to that of earlier authors. DeForest's [17] approach to a relativistic sum rule is similar to that of RCSR-I, both formally and numerically. The main difference is, in the language of Section 5, his replacement of $r_\sigma(\mathbf{0}, \mathbf{q})$ in (5.1) with the Lorentz invariant factor $\bar{G}_E^2(Q^2)/G_E^2(Q^2) = (1+\tau)/(1+2\tau)$. However, a careful inspection of the recoil factor (4.14) for a nucleon with arbitrary \mathbf{p} shows that $r_\sigma(\mathbf{p}, \mathbf{q}) \sim L_{00}(\mathbf{p}, q)/4E_{\mathbf{p}+\mathbf{q}}E_{\mathbf{p}}$

is *not* a Lorentz scalar, and that anomalous magnetic moment effects *must* enter when $\mathbf{p} \neq 0$. Hence there is, in fact, no formal justification for writing the recoil correction at $\mathbf{p} = 0$ in scalar form. It is interesting that the factor $(1+\tau)/(1+2\tau)$ is numerically nearly identical (for all $|\mathbf{p}| \sim p_F$) to our recoil factor $r_I(\mathbf{q}) \equiv r_p(\mathbf{0}, \mathbf{q}) = (E_{\mathbf{q}} + M)/2E_{\mathbf{q}}$. However, as we have seen in Figure 1, the correction due to Fermi motion, which is large due to the effects of anomalous magnetic moments, must be included for a reliable evaluation of the one-body term. For example, for $|\mathbf{q}| \sim 1$ GeV/c, the factor given by DeForest *underestimates* relativistic recoil corrections by $\sim 7\%$, compared to RCSR-II; therefore, division of the experimental data (as done in Ref. [18], for example) by DeForest's factor in the calculation of the experimental Coulomb sum will lead to an *overenhancement* of the sum by $\sim 7\%$.

The goal of Donnelly *et al.* [16] is somewhat different: to find a factor $g(\omega, \mathbf{q})$ which would replace $G_{E,p}^2(Q^2)$ in the definition of the spacelike Coulomb sum function, such that the sum will reach Z in the high- $|\mathbf{q}|$ limit. Although their approach can, in principle, achieve that goal to essentially any order of accuracy, it necessarily leads to a correction factor $g(\omega, \mathbf{q})$ which depends explicitly on the excitation energy ω . However, unless the ω -dependence can be extracted from the Coulomb response function *before* integration over ω , one will not arrive at a *non-energy-weighted* sum rule. In particular, the approach taken by Donnelly *et al.* will necessarily involve energy-weighted sum rules, which differ formally from the NRCSR, and the RCSR we have derived, in that dynamical information becomes mixed with the correlation information.

A crucial step in the derivation of our sum rule is the extraction of the energy dependence from the response function *before* integration over ω . To do this, we needed two assumptions: first, that the off-shell form factors could be obtained from the on-shell form factors $G_E(Q^2)$ and $G_M(Q^2)$ as in (2.10). Although plausible, this requires theoretical justification, i.e., a theory of the electromagnetic structure of nucleons. The second assumption is that the form factors are all proportional, as in (2.11)–(2.14). This is probably sufficiently accurate, except possibly for taking $G_{E,n} = 0$, as noted earlier. A better approximation would be to take $G_{E,n} \propto G_{E,p}$, using data from larger Q^2 , although this is incorrect near $Q^2 = 0$.

The results of this work do have some bearing on the question of the saturation of the NRCSR, which we mentioned in Section 1. The Coulomb sum function usually extracted from (e, e') experiments is that of (1.5), but with a finite upper limit on ω ; since the theoretical limit at fixed $|\mathbf{q}|$ is $\omega < |\mathbf{q}|$, this is essentially identical to (4.3). For the relativistic sum rules given in Section 5, one must divide the sum by the relativistic recoil factor $r(\mathbf{q})$, i.e., either $r_I(\mathbf{q})$ or $r_{II}(\mathbf{q})$, to obtain a form which should “saturate” at Z in the limit that $|\mathbf{q}| \rightarrow \infty$. Since $r^{-1}(\mathbf{q}) > 1$, this correction will *enhance* the experimental sum, e.g., as was

shown in Figures 2–4. At the energies of interest, however, this is only a small effect, e.g., at $|\mathbf{q}|=500$ MeV/c we have $r_I \simeq .94$ and $r_{II} \simeq .97$. It is interesting to note that these factors grow with increasing $\langle \mathbf{p}^2 \rangle$, and therefore do show *in part* the increased suppression seen in larger nuclei. Of course, many other effects may also contribute to the observed suppression.

Although we have worked within a “nucleons-only” approximation, in principle, $\bar{N} \rightarrow \bar{N}$ processes are easily included in this formalism (since the matrix element of Γ_μ between \bar{N} states is the same as that between N states). However, we do not pursue this here because other \bar{N} contributions, e.g., $N\bar{N}$ pair production, which may also enter the spacelike response, are likely to be of the same order. In order to understand fully the contribution of antinucleons to electron scattering, it will be necessary to study the spacelike response function in interacting nuclear models, and devise an appropriate renormalization scheme to obtain finite results.

Two issues which are not treated in this paper must also be addressed if the relativistic Coulomb sum rule given here is to be used for the extraction of nucleon-nucleon correlation information: namely, the contribution of meson exchange currents, and internal excitations of the nucleon. The first can be thought of as generating two-body (or more) terms in the nuclear current operator $\hat{J}_\mu(q)$ in (2.6), as done, for example in Ref. [11], where meson current contributions are actually calculated in a model. From the point of view of the sum rule, these many-body contributions can be grouped with the two-body term $\Sigma^{(2)}(\mathbf{q})$ of (4.15), to be determined *experimentally* as what is left over when $\Sigma^{(1)}(\mathbf{q})$ is subtracted from the Coulomb sum. The second issue can be thought of, to a first approximation, as the problem of removing the “background” of $N(e, e')N^*$ excitations from the (e, e') response, to leave the “nucleons-only” response, to which our relativistic Coulomb sum rules apply. This could be done theoretically by calculating the N^* background, e.g., in a noninteracting-nucleus model. Alternatively, some partially exclusive experimental information could help: for example, $(e, e'p)$ data in the quasi-free peak region can be used to normalize the knockout of protons (as opposed to N^*), although other process (e.g., $\Delta \rightarrow p+\pi$) will also contribute to this process. These two problems, and the problem of antinucleons, are subjects for future investigation.

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A Single-fermion Solutions

The Dirac equation for free point nucleons represented by the wavefunction $\psi(x)$ can be written

$$(i\gamma^\mu \partial_\mu - M)\psi(x) = 0, \quad (\text{A1})$$

where $x^\mu = (t, \mathbf{x})$. For a static, spatially uniform system, the solutions to (A1) can be expressed as momentum eigenstates with momentum \mathbf{p} . The energy eigenvalues for positive and negative energy solutions are $\epsilon_{\mathbf{p}}^{(\pm)} = \pm E_{\mathbf{p}}$, where $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + M^2}$.

From the complete set of solutions to (A1), it is possible to construct a local field operator in terms of nucleon and antinucleon degrees of freedom:

$$\hat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}s} \left[\frac{u_s(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} a_{\mathbf{p}s} + \frac{v_s(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}} e^{-i\mathbf{p}\cdot\mathbf{x}} b_{\mathbf{p}s}^\dagger \right], \quad (\text{A2})$$

where V represents the normalization volume. The nucleon and antinucleon spinors are defined

$$u_s(\mathbf{p}) = \sqrt{E_{\mathbf{p}} + M} \begin{bmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_{\mathbf{p}}+M} \chi_s \end{bmatrix} \quad v_s(\mathbf{p}) = \sqrt{E_{\mathbf{p}} + M} \begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_{\mathbf{p}}+M} \chi_s \\ \chi_s \end{bmatrix} \quad (\text{A3})$$

respectively, and have been normalized to $2E_{\mathbf{p}}$ particles per unit volume. Here χ_s is the usual two-dimensional Pauli spinor, and $a_{\mathbf{p}s}$ and $b_{\mathbf{p}s}^\dagger$ are destruction and creation operators for nucleon and antinucleons, respectively.

It is possible to generalize this formalism to include isospin degrees of freedom. We simply define $u_{s\sigma}(\mathbf{p}) \equiv u_s(\mathbf{p})\eta_\sigma$, where η_σ is a two dimensional spinor corresponding to isospin projection σ , and include a sum over σ in (A2).

B Coefficients of Moment Expansion

The expansion coefficients appearing in (5.9) are:

$$r_0^p = \frac{E_{\mathbf{q}} + M}{2E_{\mathbf{q}}} \quad (\text{B1})$$

$$r_1^p = \frac{\mathbf{q}^2}{2ME_{\mathbf{q}}^3} \quad (\text{B2})$$

$$r_2^p = \frac{-\mathbf{q}^4(2E_{\mathbf{q}} + M)}{4M^2 E_{\mathbf{q}}^3(E_{\mathbf{q}} + M)^2} + \frac{(1 + \kappa_p)^2 \mathbf{q}^2}{2M^2 E_{\mathbf{q}}(E_{\mathbf{q}} + M)} \quad (\text{B3})$$

$$r_3^p = \frac{2E_{\mathbf{q}}^4 - M(E_{\mathbf{q}} + M)(2E_{\mathbf{q}}^2 - 3M^2)}{4M^2 E_{\mathbf{q}}^5(E_{\mathbf{q}} + M)} - \frac{(1 + \kappa_p)^2}{2M^2 E_{\mathbf{q}}(E_{\mathbf{q}} + M)} \quad (\text{B4})$$

$$r_2^n = \frac{\kappa_n^2 \mathbf{q}^2}{2M^2 E_{\mathbf{q}}(E_{\mathbf{q}} + M)} \quad (\text{B5})$$

$$r_3^n = \frac{-\kappa_n^2}{2M^2 E_{\mathbf{q}}(E_{\mathbf{q}} + M)} \quad (\text{B6})$$

We have $r_0^n = r_1^n = 0$ by our approximation $G_{E,n}(Q^2) = 0$.

Expressions (B5) and (B6) can be obtained from (B3) and (B4), respectively, by keeping only terms proportional to $(1 + \kappa_p)^2$ and letting $(1 + \kappa_p)^2 \rightarrow \kappa_n^2$. The corresponding expressions for Dirac nucleons are obtained by setting $\kappa_p = \kappa_n = 0$.

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Figure 1: Recoil correction factors vs. three-momentum transfer \mathbf{q} . The curves representing $r_I(\mathbf{q})$ and $r_{II}(\mathbf{q})$ with $\kappa_p = \kappa_n = 0$ are nearly indistinguishable on this scale.

Figure 2: Relativistic Coulomb sum (divided by Z) vs. three-momentum transfer \mathbf{q} in the Fermi gas model, ignoring anomalous magnetic moments. The dashed curve is $S_I(\mathbf{q})/Z$, and the solid curve is $S_{II}(\mathbf{q})/Z$. For comparison, the dot-dashed curve is the unmodified sum $\Sigma(\mathbf{q})/Z$.

Figure 3: Relativistic Coulomb sum (divided by Z) vs. three-momentum transfer \mathbf{q} in the Fermi gas model, including the proton anomalous magnetic moment. The dashed curve is $S_I(\mathbf{q})/Z$, and the solid curve is $S_{II}(\mathbf{q})/Z$. For comparison, the dot-dashed curve is the unmodified sum $\Sigma(\mathbf{q})/Z$.

Figure 4: Relativistic Coulomb sum (divided by Z) vs. three-momentum transfer \mathbf{q} in the Fermi gas model, including proton and neutron anomalous magnetic moments. The dashed curve is $S_I(\mathbf{q})/Z$, and the solid curve is $S_{II}(\mathbf{q})/Z$. For comparison, the dot-dashed curve is the unmodified sum $\Sigma(\mathbf{q})/Z$.

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